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A DECISION THEORETIC APPROACH TO LANCHESTER PROCESSES. I. THE O--ETC(U)
APR 78 P P PERLA N00014-77-C-0161

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A Decision Theoretic Approach to Lanchester Processes. I. The One Period Problem.

by

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Technical Report No. 146
April, 1978
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*Research supported in part by contract N00014-77-C-0161 (NR 277-227) from the Office of Naval Research.

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Abstract

This paper develops a decision theoretic framework for one period or "fight-to-the finish," stochastic Lanchester models. Costs are assigned to the employment and destruction of friendly forces and rewards granted for "victory." Solutions for certain one period problems are obtained approximately by using the standard central limit theorem or by making use of certain martingale central limit theorems. Some examples are presented.

1. Introduction

The military operations research literature abounds with works dealing with Lanchester type attrition models (the reader is referred to the tutorial by Taylor [11] for a survey). Unfortunately, the use of these models in the development of optimal decision making remains a relatively unexplored area of analysis. Problems of optimal fire allocation as differential games such as those addressed by Isbell and Marlow [2], Weiss [13], Kawara [4], and Taylor [9], [10] involve decision making, but the research in this area is somewhat limited in scope and suffers from its reliance on deterministic rather than stochastic models. Furthermore, models of decision making have not been built on a decision theoretic structure, one which involves an assignment of costs to the employment and destruction of friendly units and rewards for victory and the destruction of enemy units.

This paper is concerned with one period or "fight-to-the-finish" Lanchester models. A decision theoretic framework suitable for such problems is developed. This framework is especially simple in that it allows for only one type of decision, the choice of initial force level. This approach can be generalized to handle multi-period or sequential decision models. Such models allow for complex decision-making such as the calling in of reinforcements or withdrawal of committed units as well as allowing for changes in tactics during the conflict. Such generalizations are reserved for a later paper.

The evolution of the attrition process is assumed to be described by a fixed stochastic Lanchester model. The determination

of an optimal decision requires the calculation of quantities such as the probability of victory for some side and the expected number of survivors conditional upon victory. For stochastic Lanchester models, these quantities are not available in a convenient closed form. The difficulty of obtaining any expressions for these quantities, even ones which are far too complex to be of any utility, has retarded the development of any strategic planning based upon Lanchester-type attrition structures. Instead of closed form expressions, we employ several approximation techniques. These techniques are based on both the classical central limit theorem and more recent martingale central limit theorems (see Watson [12] and Perla [5] for a full description of the latter methods). The approximations themselves are still complicated and the optimal decisions described in this paper must be determined numerically.

2. Decision Theoretic Framework

We concentrate on the basic one-stage decision framework which is based on a very simple military situation. The decision maker must decide whether to accept a combat action, and, if so, the amount of force to employ. This decision will be based on a number of objective and subjective assessments such as the type of conflict involved, the strength of the opponents, the costs of employing units, the forces available, and the relative reward for victory.

The mathematical formulation of this decision problem is constructed along the following lines. First, we define an underlying space Θ of possible initial conditions associated with the particular situation and beyond the control of the decision maker. This may include such factors as the enemy force level, the nature of the conflict to be fought, and the effectiveness of the opponents forces. Any of these quantities may be known or unknown to the decision maker. In the case that some component, the opponent's force level for example, were unknown, the decision maker may have prior information which can be expressed in the form of a prior distribution on Θ. We next define the space of possible battle outcomes, Ω. For our situation an outcome consists of a pair which indicates the victory and the number of survivors. Finally, there is a space of available decisions, D. In this simple framework we allow only the choice of initial force levels; however, this could easily be extended to include tactical considerations. A real-valued loss function L is defined on $\Omega \times D$

with $L(\omega,\delta)$ representing the loss to the decision maker when outcome $\omega \in \Omega$ occurs given that he has made the decision $\delta \in D$.

Furthermore, if $P(\cdot|\delta,\theta)$ is a probability distribution on Ω for each $\delta \in D$ and $\theta \in \Theta$ and $F(\cdot)$ a probability distribution on Θ, then

$$P(\cdot|\delta) = \int_{\Theta} P(\cdot|\delta,\theta)dF(\theta)$$

is a probability distribution on Ω for each $\delta \in D$. We will also write $P(\cdot|\delta)$ as $P_\delta(\cdot)$.

The expected loss, or risk, of any decision $\delta \in D$ is defined by

$$\rho(\delta) = \int_{\Omega} L(\omega,\delta)dP_\delta(\omega).$$

The decision maker wishes to choose a decision $\delta^* \in D$ such that $\rho(\delta^*)$ is a minimum.

In order to analyze the combat decision problem as outlined above, it is important to understand the character and role of each of its components.

The space Θ of initial conditions includes those elements of the conflict situation beyond the immediate control or influence of the decision maker. Important factors which might be represented by elements of Θ are the numerical strength of hostile forces and their combat power, as quantified by their attrition coefficients, as well as the effects of terrain and weather. We assume that the opponent's force level, while possibly unknown to the decision maker, is fixed a priori. It will also be assumed that the dynamics

of the combat, represented by the mathematical attrition model used to describe it, is also beyond the decision maker's control and may well be unknown. This uncertainty requires the commander to formulate an a prior probability distribution F for the possible states in Θ based on the information available to the commander concerning the unknown quantities and his interpretation of that information. Thus Θ represents the underlying structure of the combat situation which may be possible, and F the commander's uncertainty about that structure. In the remainder of this paper full information will be assumed; that is, the commander's prior places a probability mass of one on some particular element of Θ. The general case will be discussed in more detail in a subsequent paper.

In the basic decision problem, conflict continues until the force level of one side is reduced to zero. Since the elements of the outcome space Ω represent the final state of the conflict, or terminal point, they may be expressed in the form $(X,0)$ or $(0,Y)$ where X or Y is the number of survivors of the victorious side. More general termination criteria will not be discussed here.

Decisions are restricted to determination of the number of units to employ, and all such units are committed at the beginning of the conflict. The basic problem addressed here does not allow for possible reinforcements and is treated as a one stage decision problem.

The loss function $L(\omega,\delta)$, and the cost and reward structure associated with it, are constructed in terms of a basic unit of value defined as the cost of the destruction of a single friendly

unit. The costs for troop employment and the reward for victory are measured in terms of this unit of value. Furthermore, it will be assumed that partial destruction of the enemy force is of no value. Under these sorts of assumptions the loss function may be written

$$L(\omega,\delta) = cX_o(\delta) + [X_o(\delta) - X_f(\omega)] - VI(\omega) \tag{1}$$

where $X_o(\delta)$ is the initial friendly force level chosen by decision $\delta \in D$, $X_f(\omega)$ is the surviving friendly force level specified by outcome $\omega \subset \Omega$, and $I(\omega) = 1$ if $X_f(\omega) > 0$ and $I(\omega) = 0$ if $X_f(\omega) = 0$. The constants c and V represent the cost of employing troops and the value of victory respectively. Note that in this case, the value of victory does not depend on the number of friendly survivors as long as there is at least one. Alternative formulations in which the surviving force level plays a more direct role in assessing the value of victory will not be discussed here.

Solution of the decision problem requires a knowledge of the expected value of the friendly force level at the conclusion of the combat and the probability of a friendly victory. These quantities are calculated from the probability distributions $P(\cdot|\delta,\theta)$, derived from the stochastic Lanchester model appropriate to the state θ. The usual stochastic models of the Lanchester-type are in the form of bivariate or multivariate Markov Chains, and these are the types of models which will be employed in the sequel.

Markov chains are characterized first of all by a state space, the elements of which represent the state or condition of the process at any point in time. In stochastic combat models, the

state space is generally a cartesian product of the form $\otimes \underset{\sim}{N}$ where $\underset{\sim}{N} = (1,2,...N)$ for some sufficiently large integer N. The dimension of the space depends on the number of distinct types of units available on either side, each component representing the number of surviving units of a particular type. In the simple case as it has been outlined above, there is only one type of unit on each side and so the state space, E, is simply $\underset{\sim}{N} \times \underset{\sim}{N}$. The elements of the state space represent the surviving force level on each side, and are thus ordered pairs of the form (X,Y).

Since we are concerned with the distribution of survivors at the conclusion of the combat and not at any specified time during its course, we may consider the combat process a discrete one, with epochs marked by the occurrence of a casualty. In this case, transitions are made from a state (X,Y) to either of the states (X-1,Y) or (X,Y-1). In general, the transition probabilities may be written:

$$P[(X,Y),(X-1,Y)] = \frac{f(X,Y)}{f(X,Y)+g(X,Y)}$$

$$P[(X,Y),(X,Y-1)] = \frac{g(X,Y)}{f(X,Y)+g(X,Y)} ,$$

for suitable functions f and g. The forms of the f and g functions are derived from the appropriate physical assumptions about the combat process. (See Karr [3] for example).

From the transition probabilities it is possible to calculate the distribution of final configurations. This fact, in turn, allows calculation of winning probabilities and expected number of survivors, thus providing the tools to solve the one stage decision problem. An example is presented in the next section.

3. An Example of the Basic Decision Problem

As an example of the basic decision problem consider the stochastic version of the Lanchester Linear Law. The process is of the form $\{(X_n, Y_n)\}$ where X_n and Y_n are the force levels after a total of n casualties. If the initial force levels are X_o and Y_o respectively, then $X_n + Y_n + n = X_o + Y_o$.

If we define $\Delta X_n = X_{n+1} - X_n$ and ΔY_n similarly, then the transition probabilities of the process may be expressed in the following form:

$$P[\Delta X_n = -1, \Delta Y_n = 0 \mid (X_n, Y_n)] = \frac{a}{a+b} = q$$

$$P[\Delta X_n = 0, \Delta Y_n = -1 \mid (X_n, Y_n)] = \frac{b}{a+b} = p.$$

The above holds for $X_n, Y_n > 0$. The constants a and b are characteristic of the attrition process and $p + q = 1$.

As can be seen, the transition probabilities for the model are state independent provided both sides have survivors. States in which either force level has dropped to zero are absorbing states and no further transitions are possible. Thus the elements $\omega \in \Omega$ are of the form $(X,0)$ or $(0,Y)$ with $X \leq X_o$, $Y \leq Y_o$. In this case the combat process takes on the form of a restricted random walk in the plane, beginning at the point (X_o, Y_o) with steps either to the left or downward. (See Figure 1).

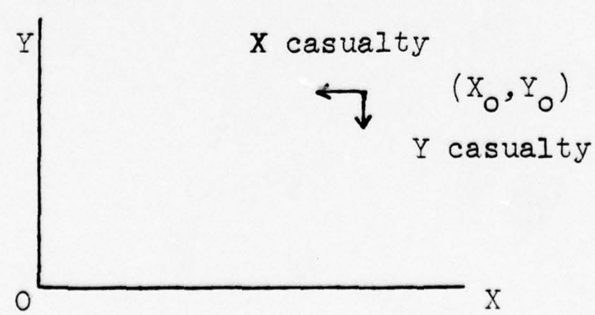


Figure 1

The X axis and Y axis are absorbing barriers for this walk, and the probability $P(\omega|\delta,\theta)$ may be written in terms of the negative binomial distribution.

Thus, if $\theta \in \Theta$ specifies a linear attrition model of this form, with initial Y force level Y_o and attrition parameters a and b, and if $\delta \in D$ specifies an initial X force level of X_o, then the probability of $\omega \in \Omega$, where ω specifies that X wins with X_f survivors is given by:

$$P(\omega|\delta) = P[(X_f,0)|X_o,Y_o,p] = \binom{X_o - X_f + Y_o - 1}{X_o - X_f} p^{Y_o} q^{X_o - X_f} \; . \qquad 1 \leq X_f \leq X_o$$

$$= \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c} \; , \qquad 0 \leq X_c \leq X_o - 1 \tag{2}$$

where $X_c = X_o - X_f$ represents the X casualties.

Define the set $\Omega_{X_o} \subset \Omega$ by

$$\Omega_{X_o} = \{\omega \in \Omega | \omega = (X,0), 1 \leq X \leq X_o\},$$

that is the points on the positive X axis to the left of the initial X force level (excluding the origin). Then the probability of an X victory under the conditions assumed above is

$$P(\Omega_{X_o}|X_o,Y_o,p) = \sum_{X_c=0}^{X_o-1} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c} \; .$$

Thus for a Y_o and ρ fixed by $\theta \in \Theta$, the risk of decision $\delta = X_o$ is given by

$$\rho(X_o) = cX_o + E[X_c | X_o, Y_o, p] - VP[\Omega_{X_o} | X_o, Y_o, p]$$

$$= cX_o + E[X_c | \Omega_{X_o}, X_o, Y_o, p] P[\Omega_{X_o} | X_o, Y_o, p] \tag{3}$$

$$+ E[X_c | \Omega^c_{X_o}, X_o, Y_o, p] P[\Omega^c_{X_o} | X_o, Y_o, p] - VP[\Omega_{X_o} | X_o, Y_o, p].$$

Employing the correct negative binomial expressions in equation (3) we have

$$\rho(X_o) = cX_o + \sum_{X_c=0}^{X_o-1} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c}$$

$$+ X_o \left[1 - \sum_{X_c=0}^{X_o-1} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c} \right] \tag{4}$$

$$- V \sum_{X_c=0}^{X_o-1} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c}$$

or

$$\rho(X_o) = cX_o + \sum_{X_c=0}^{X_o-1} X_c \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c}$$

$$+ X_o \sum_{X_c=X_o}^{\infty} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c} \tag{5}$$

$$- V \sum_{X_c=0}^{X_o-1} \binom{X_c + Y_o - 1}{X_c} p^{Y_o} q^{X_c} .$$

The form of expression (5) provides little insight into the qualitative behavior of the risk function. In order to solve the decision problem, the value of X_o which minimizes (5) must be obtained. Again, the complexity of the expression renders this task somewhat difficult, requiring an extensive numerical search. Thus the risk function of even this, the simplest of the Lanchester models, presents some serious obstacles to the ready solution of the one-stage decision problem.

For more complex models, such as that based on the Lanchester Square Law, the mathematical difficulties are compounded; even basic probabilities such as $P[(X_f,0)|X_o,Y_o,p]$ involve summations of some complexity (see Smith [8]). One attempt to circumvent the intractability of expression (5) is presented in Section 4. A more general method of attack suitable for a variety of Lanchester models is presented in Section 5.

4. Approximate Solution of the One-Stage Problem

The discussion of the one-stage decision problem as presented in Section 3 revealed the difficulty of solving for the decision, $X_o(\delta)$, which minimizes the risk function ρ. Typically, closed form solutions of these optimization problems cannot be found due to the complexity and mathematical intractability of the required expressions. This failure of analytic methods leads to a consideration of techniques of approximation, in order to simplify the expressions with which we must deal, and also of numerical methods which may be employed to solve the problem.

One likely approach is to employ a central limit theorem to approximate cumbersome probability distributions by the more familiar and well studied normal distribution. Consider the Linear Law example presented above. In particular, consider the term

$$\sum_{X_c=0}^{X_o-1} \binom{X_c+Y_o-1}{X_c} p^{Y_o} q^{X_c}$$

in expression (5). This term represents the probability that the X side is victorious conditional on X_o, Y_o and p. This probability is merely the probability that a negative binomial random variable is less than X_o. The standard central limit theorem applies, allowing the approximation of this sum by the appropriate value of the standard normal cumulative distribution function Φ, under the conditions that X_o and Y_o are large (say larger than 30) and p is not very extreme. In combat models, both of the latter assumptions are generally valid.

In this manner, expression (5) may be written in an approximate form as

$$\rho(X_o) \cong cX_o + \frac{qY_o}{p}\,\Phi\left\{\frac{pX_o - qY_o - 1}{(qY_o)^{1/2}}\right\} + X_o\left\{1 - \Phi\,\frac{pX_o - qY_o}{(qY_o)^{1/2}}\right\} - V\Phi\left\{\frac{pX_o - qY_o}{(qY_o)^{1/2}}\right\}. \tag{6}$$

We simplify expression (6) even further by assuming the values of X_o, Y_o and p are such that we may consider the arguments of all three normal distribution functions in the above expression to be the same without serious loss in accuracy. (Note that for this reason the usual continuity correction will be ignored as well.) This leads to the approximation

$$\rho(X_o) \cong (c+1)X_o - \Phi\left[\frac{pX_o - qY_o}{(qY_o)^{1/2}}\right]\left\{X_o + V - \frac{qY_o}{p}\right\} . \tag{7}$$

The optimal decision, X_o^*, is obtained from expression (7) by differentiating it as a function of X_o, (treated as a continuous rather than an integer variable), setting the derivative equal to zero, and solving for X_o. If we let $\varphi(x)$ be the normal probability density function ($\varphi(x) = \Phi'(x)$) then we may write

$$\rho'(X_o) = c + 1 - \Phi\left[\frac{pX_o - qY_o}{(qY_o)^{1/2}}\right] - \left[\frac{p(X_o + V) - qY_o}{(qY_o)^{1/2}}\right]\left[\varphi\left(\frac{pX_o - qY_o}{(qY_o)^{1/2}}\right)\right]. \tag{8}$$

The continued presence of the normal distribution function, Φ, remains something of a problem; however, assuming its argument is

sufficiently large, we may approximate it through use of the Mills ratio technique (see Feller [1], p.166). This approximation allows the estimation of normal tail probabilities by the ratio of the normal pdf to its argument:

$$\text{for } x \gg 0,\ 1 - \Phi(x) \sim \varphi(x)/x.$$

The use of this approximation requires the consideration of two cases.

Case 1: $pX_0 - qY_0 > 0$ (the subscript 0 is deleted in the sequel). In this case we approximate

$$1 - \Phi\left[\frac{pX - qY}{(qY)^{1/2}}\right] \approx \frac{(qY)^{1/2}}{pX - qY}\,\varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right).$$

Thus

$$\begin{aligned} \rho'(X) &\simeq c + \frac{(qY)^{1/2}}{pX - qY}\,\varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right) - \left[\frac{pX - qY + pV}{(qY)^{1/2}}\right]\varphi\left[\frac{pX - qY}{(qY)^{1/2}}\right] \\ &= c - \varphi\left(\frac{pX - qY}{(qY)^{1/2}}\right)\left\{\frac{pX - qY}{(qY)^{1/2}} + \frac{pV}{(qY)^{1/2}} - \frac{(qY)^{1/2}}{pX - qY}\right\}. \end{aligned} \tag{9}$$

Since X and Y are assumed to be large, we consider the term $\frac{(qY)^{1/2}}{pX - qY}$ to be negligible. This gives

$$\rho'(X) \simeq c - \varphi\left[\frac{pX - qY}{(qY)^{1/2}}\right]\left\{\frac{pX - qY}{(qY)^{1/2}} + \frac{pV}{(qY)^{1/2}}\right\}. \tag{10}$$

Defining $\frac{pX - qY}{(qY)^{1/2}} = \eta$ and $\frac{pV}{(qY)^{1/2}} = \alpha$ we have

$$\rho'(X) = f(\eta) = c - \varphi(\eta)(\eta + \alpha). \tag{11}$$

In order to solve for the optimal value of X, we set $f(\eta)$ to zero and solve for η, that is, find those values of η such that

$$\varphi(\eta) = \frac{c}{\eta+\alpha} \tag{12}$$

where c and α are known constants and η is assumed to be positive by definition.

The solutions of (12) are the intersections of the standard normal density function with a certain hyperbola. Only positive roots need to be considered. The exact characterization of the roots is difficult and ultimately (12) must be solved numerically. Nevertheless some insights can be gained from a qualitative examination of (12).

There are two cases, $c/\alpha < (2\pi)^{-1/2}$ and $c/\alpha \geq (2\pi)^{-1/2}$. In the first case, $c/\alpha < (2\pi)^{-1/2}$, there is exactly one positive solution of (12). Figure 2 provides the approximate diagram for this case. If η_0 is the negative root, we wish to know for

what value x will $\eta_0 + x$ also be a root, clearly $\eta_0 + x > 0$. Given $\varphi(\eta_0) = c/(\eta_0+\alpha)$ and $\varphi(\eta_0+x) = c/(\eta_0+x+\alpha)$, one can easily find that x must satisfy

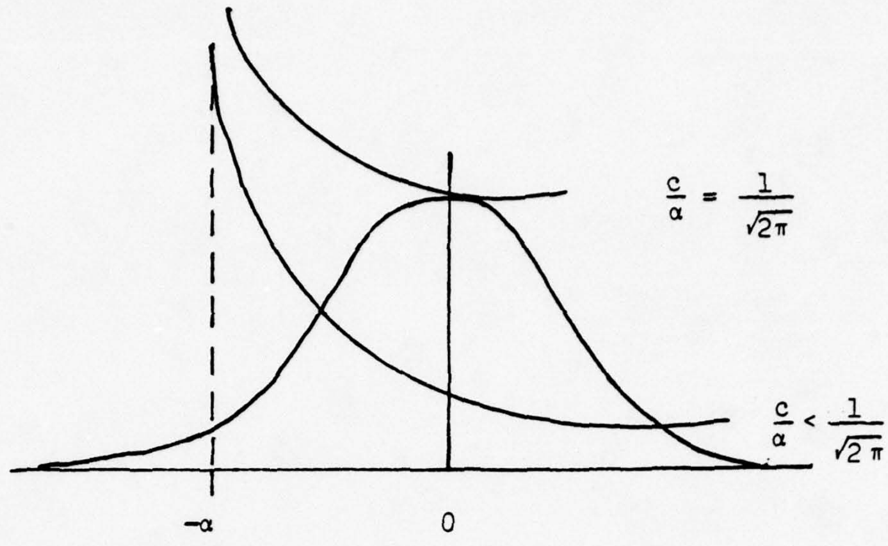
$$\exp(\eta_0 x + \tfrac{1}{2}x^2) = 1 + x/(\eta_0+\alpha). \tag{13}$$

The equation has a solution if $x = 0$. The left side has a negative derivative at 0 but the second derivative is positive. Thus $\exp(\eta_0 x + \frac{1}{2}x^2)$ will cross the line $1 + x/(\eta_0+\alpha)$ exactly once for $x > 0$. This crossing point when added to η_0 gives the solution to (12).

The situation is not clear if $c/\alpha \geq (2\pi)^{-1/2}$. There may be none, one, or two roots. The ratio c/α is not sufficient to determine which.

Now suppose $c/\alpha > \frac{1}{\sqrt{2\pi}}$. In this case, there can be no negative η solutions to equation (12). However, it can be shown by the same argument employed above that there can be at most two solutions to (10) when $pX - qY > 0$. A single solution is also possible. It can be shown that if the smallest solution η_0 is such that $\eta_0 > \frac{1}{\eta_0 + \alpha}$, it is, in fact, the unique positive solution to (12).

In summary, if $pX - qY > 0$, there are at most two admissible solutions to equation (12), that is, at most two critical values for the risk function as a function of the X force level.


$\frac{c}{\alpha} = \frac{1}{\sqrt{2\pi}}$
$\frac{c}{\alpha} < \frac{1}{\sqrt{2\pi}}$
$-\alpha$
0

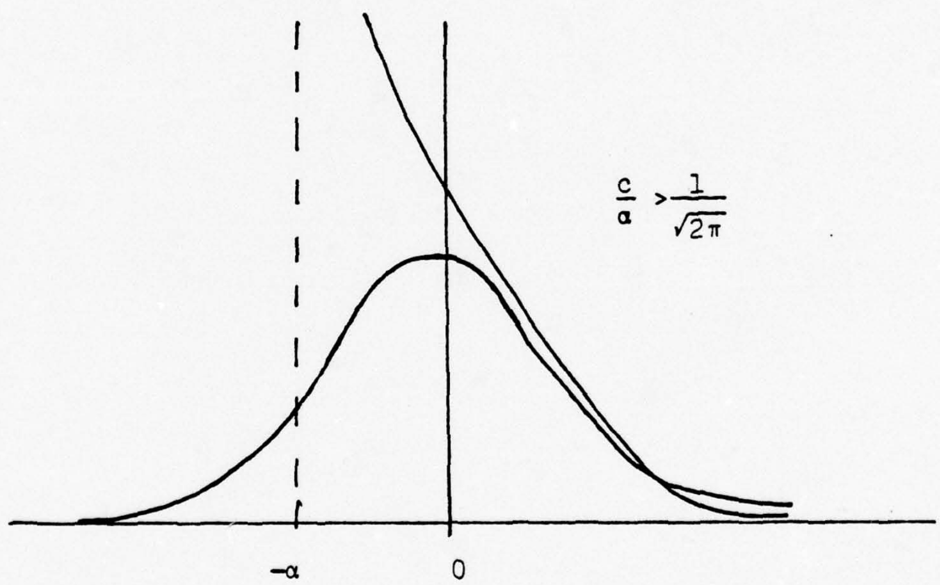

$\frac{c}{\alpha} > \frac{1}{\sqrt{2\pi}}$
$-\alpha$
0

Figure 2

Case 2: $pX - qY < 0$. The solution of the problem for this case follows the same lines as that for Case 1. In this manner we arrive at the equation

$$\varphi(\eta) = \frac{c+1}{\eta+\alpha} \tag{14}$$

where η and α are defined as before but with the restriction that only negative solutions of (14) are valid. Equation (14) is of the same form as (12), and so it follows that there is at most one valid solution to it, and such a solution can only exist if $\frac{c+1}{\alpha} < \frac{1}{\sqrt{2\pi}}$.

In this latter case, $c/\alpha < \frac{1}{\sqrt{2\pi}}$ (since α is positive). Thus it is possible to have critical values X_1 and X_2 for the risk function such that $pX_1 - qY < 0$ and $pX_2 - qY > 0$. Note also that if $c/\alpha > \frac{1}{\sqrt{2\pi}}$, then $(c+1)/\alpha > 1/(2\pi)^{1/2}$, and so there are at most two critical X values for the risk function.

A further examination of the character of the risk function reveals that there is a point at which the cost of the forces employed is equal to or just greater than the value of victory. As the initial force increases beyond that point, the large cost of the manpower employed dominates the value of victory, thus the loss and risk increase to infinity as the force level goes to infinity. If there are only two critical points for the risk function, the larger of the two must represent a local minimum or a saddle point.

Furthermore, for very large values of c, it is possible that there are no valid solutions to equations (12) and (14).

In this case, the risk function has no extreme values except for the boundary value at zero. (That is, the cost of employing troops is so high relative to the value of victory that the combat is best avoided.)

Summarizing the results of the above analysis, the risk function can have zero, one, or two critical values. These findings coincide with intuitive conjectures of possible reasonable shapes for the risk function based on the proposed loss and reward structure. In general, the risk function can be expected to exhibit one of the qualitative types of behavior exhibited in Figure 3.

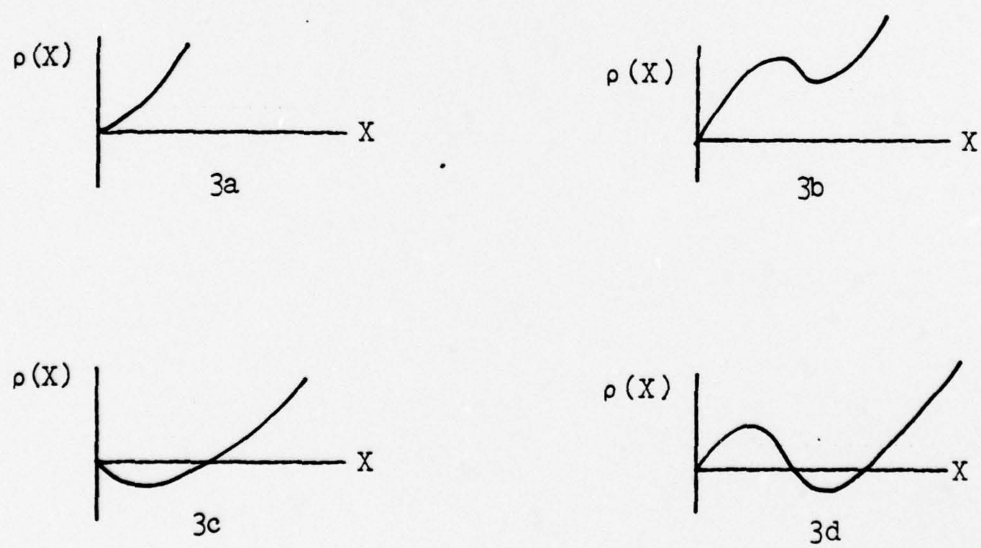


Figure 3

The difference between functions of the form (3a) and (3b) has little impact on decision making. In either of these cases, the optimal value of the risk function occurs for $X = 0$ and no manpower should be committed to the combat. In the case of (3c) and (3d), however, true optimal force levels exist and correspond to positive X values.

In order to determine the optimal value of X in these latter cases, it is necessary to solve (12) or (14) numerically. Standard numerical techniques such as the bisection method are applicable and proved to give results which are intuitively reasonable and appealing in a minimal amount of computing time. Some specific examples were considered, and selected results are included in Table 1. It is important to notice that the optimal force levels are substantially higher than the minimum force level required for victory by the deterministic model. This is not surprising in view of the fact that the deterministic analysis does not take any account of the relative cost of employing units or the value of victory nor does it recognize the uncertainty of the conflict's outcome. It also highlights the importance of using a stochastic analysis to evaluate military requirements.

Although the basic approach that has been described in this section is relatively straightforward and appears to give reasonable results, it does have some drawbacks. Most serious is the fact that this particular approach is not readily applicable to other stochastic Lanchester type models in which the transition probabilities are state dependent. In these cases, the complexity

of the expressions for the required probabilities makes an immediate application of standard central limit theorems difficult. One highly effective alternate technique suggested by Watson [12] is discussed in the next section.

Table 1 - Numerical Results for the One Stage Decision Problem Lanchester Linear Law, Standard Central Limit Theorem Approach

Notation: Y_o - Initial hostile force level

p - Probability next casualty is an enemy $(p = \frac{b}{a+b})$

c - Cost of employing friendly troops

V - Reward for destroying entire enemy force

X_o - Optimal friendly force level, central limit approach

ρ_o - Risk of optimal force

X_L - Minimal force required to win battle (enemy destroyed with at least one friendly survivor) from deterministic equations.

Y_o	p	c	V	X_o	ρ_o	X_L
100	0.5	0.5	500	136.94	-328.89	101
1000	0.5	0.5	5000	1134.40	-3426.12	1001
100	0.3	0.5	500	299.14	-111.95	234.3
1000	0.3	0.5	5000	2578.38	-1363.33	2334.3
100	0.7	0.5	500	64.92	-423.43	43.86
1000	0.7	0.5	5000	507.58	-4314.08	429.57
100	0.5	0.6	500	135.92	-318.45	101
1000	0.5	0.6	5000	1131.67	-3312.00	1001
100	0.3	0.6	500	296.90	-82.16	234.3
1000	0.3	0.6	5000	2572.47	-1105.37	2334.3
100	0.7	0.6	500	64.40	-416.94	43.86
1000	0.7	0.6	5000	506.16	-4263.16	429.57

5. Martingale Approximations and Decision Making

The central limit theorem method used in the last section is limited to the linear law case in which the transition probabilities are spatially homogeneous. For more complicated attrition structures, one must turn to martingale approximations originally introduced by Watson [12] and further developed by Perla [5] and Perla and Lehoczky [6]. We briefly review the method and then illustrate its use in decision making.

Suppose $\{(X_n,Y_n),n \geq 0\}$ is a discrete stochastic process based on a casualty time scale, that is, X_n and Y_n are the opposing force levels after a total of n casualties have occurred. The usual models for a combat process of this type take the form of bivariate Markov chains. As discussed above, the transition probabilities of such chains may be given in general by

$$(X_{n+1},Y_{n+1}) = \begin{cases} (X_n-1,Y_n) \text{ with probability } \dfrac{g(X_n,Y_n)}{f(X_n,Y_n)+g(X_n,Y_n)} \\ \\ (X_n,Y_n-1) \text{ with probability } \dfrac{f(X_n,Y_n)}{f(X_n,Y_n)+g(X_n,Y_n)} \end{cases}$$

for some suitable functions f and g. A discrete time martingale can be defined from this chain by finding a function $K(\cdot,\cdot)$ such that

$$K(X,Y) = [K(X-1,Y)g(X,Y) + K(X,Y-1)f(X,Y)]/[f(X,Y) + g(X,Y)]. \quad (15)$$

Equation (15) can, in general, be solved inductively for the function K. Let

$$\frac{g(X_n,Y_n)}{f(X_n,Y_n) + g(X_n,Y_n)} = q(X_n,Y_n) \text{ and } \frac{f(X_n,Y_n)}{f(X_n,Y_n) + g(X_n,Y_n)} = p(X_n,Y_n).$$

$p(X_n, Y_n)$ is the probability that the next casualty is a Y, while $q(X_n, Y_n)$ represents the probability that the next casualty is an X.) The function K can then be derived from

$$\begin{aligned} K(X-1,Y) - K(X,Y) &= -p(X,Y)\theta(X,Y) \\ K(X,Y-1) - K(X,Y) &= q(X,Y)\theta(X,Y) \end{aligned} \qquad (16)$$

where θ is some function of (X,Y) which may be chosen in a suitable manner. (See Watson [12]). Some examples of these types of martingale functions are

$$K_1(X,Y) = pX - qY \quad \text{where} \quad p = \frac{b}{a+b},\ q = \frac{a}{a+b}$$

for a Linear Law model with attrition constants a and b, or

$$K_2(X,Y) = \tfrac{1}{2}[bX(X+1) - aY(Y+1)]$$

for a Square Law model.

Thus, if a function K satisfies equation (16), the discrete stochastic process $\{(K(X_n,Y_n),\mathcal{B}_n), n \geq 0\}$ is a martingale, where $\mathcal{B}_n = \mathcal{B}(X_i, 0 \leq i \leq n)$ is the σ-field generated by $\{X_i, 0 \leq i \leq n\}$. In the casualty time scale X and Y are functionally related by $X_n + Y_n + n = X_0 + Y_0$. The σ-field generated by the (X,Y) pairs is simply that generated by either component individually.

If we let (X_f, Y_f) be the force levels when the conflict terminates, then we must find $P(X_f>0) = P(X \text{ wins})$ and $E(X_f|X_f>0)$, the expected number of survivors conditional on victory. These quantities are involved in the risk function. They can be calculated approximately by noting that $K_f = K(X_f, Y_f)$ will be approximately normally distributed with some specified mean and variance. In many cases, the distribution of $K(X_f, Y_f)$ can be found and the required quantities therefore calculated approximately.

We illustrate with the square law case where $K(x,y) = \frac{1}{2}bx(x+1) - \frac{1}{2}ay(y+1)$. The distribution of $K_f = K(X_f, Y_f)$ will be approximately normal with mean

$$\mu = \tfrac{1}{2}bX_0(X_0+1) - \tfrac{1}{2}aY_0(Y_0+1) \tag{17}$$

and variance

$$\sigma^2 = \begin{cases} \frac{1}{3}(b^2X_0^3 + a^2Y_0^3 - (8b\mu^3)^{1/2}) \text{ if } \mu > 0 \\ \frac{1}{3}(b^2X_0^3 + a^2Y_0^3 - (-8a\mu^3)^{1/2}) \text{ if } \mu < 0. \end{cases} \tag{18}$$

Now $K_f > 0$ if and only if $X_f > 0$ and $Y_f = 0$. Thus $P(X \text{ wins}) = P(X_f>0) = P(K_f>0) \approx \Phi(\mu/\sigma)$. Furthermore, if $K_f > 0$, then $K_f = 1/2\ bX_f(X_f+1)$ or $X_f = 1/2((1 + \frac{8K_f}{b})^{1/2} - 1)$. It follows that

$$E(X_f | X_f>0) \approx 1/2[\int_0^\infty (1+ \frac{8K}{b})^{1/2} \frac{1}{\sigma\sqrt{2\pi}} \exp(-1/2(\frac{x-\mu}{\sigma})^2)dx/(\Phi(\frac{\mu}{\sigma}) -1)].$$

6. Martingale Solutions for the Linear and Square Laws

The martingale techniques outlined in the previous section may be used to obtain approximate solutions to the one stage decision problem through numerical approximation and computer optimization techniques. As an example of the general methods employed, this section presents the details and results of the solution to the Linear Law case, and outlines the approach required for solution of a Square Law problem.

Let the initial force configuration be (X_0, Y_0) with X_0, Y_0 large. Define $k(x,y) = px - qy$ where $p = \frac{b}{a+b}$ and $q = 1 - p$. Let $\mu = pX_0 - qY_0$. The distribution of K_f is approximated by a normal distribution with mean μ and variance σ^2 given by

$$\sigma^2 = \begin{cases} pX_0 & \text{if} \quad \mu < 0 \\ qY_0 & \text{if} \quad \mu > 0. \end{cases}$$

Since the value of K_f is positive if and only if $X_f > 0$ and $Y_f = 0$, $P(X \text{ win}) = P(K_f > 0) \doteq \Phi(-\mu/\sigma)$ where $\Phi(x)$ is the standard normal distribution function at the point x. The expected value of X_f is obtained by the same type of argument. If $K_f > 0$, then $Y_f = 0$ and $X_f = K_f/p$. Thus

$$E[X_f | X \text{ wins}] = \frac{1}{p} E(K_f | K_f > 0) = \frac{1}{p}[\mu + \sigma\varphi(\mu/\sigma)/\Phi(\mu/\sigma)],$$

where φ is the standard normal density function. Thus

$$E[X_0 - X_f | X \text{ wins}] = X_0 - \frac{1}{p}[\mu + \sigma\varphi(\mu/\sigma)/\Phi(\mu/\sigma)] =$$

$$X_0 - \frac{pX_0}{p} + q\frac{Y_0}{p} - \frac{\sigma}{p}\frac{\varphi(\mu/\sigma)}{\Phi(\mu/\sigma)} = \frac{q}{p} Y_0 - \frac{\sigma}{p}[\varphi(\mu/\sigma)/\Phi(\mu/\sigma)].$$

The risk function, $\rho(X_0)$, then takes on the approximate form

$$\rho(X_0) \cong (c+1)X_0 - \Phi(\mu/\sigma)[v+X_0 - \frac{\sigma}{p} Y_0] - \frac{\sigma}{p}\phi(\mu/\sigma). \tag{19}$$

The solution of the one stage problem requires the minimization of (19) as a function of X_0.

It is clear from the discussion of the shape of the one stage risk function in Section IV, as well as from intuitive considerations, that the optimal value of X_0 must be to the right of the point $q/p\ Y_0$. In this case, $\mu = pX_0 - qY_0 > 0$ and so $\sigma^2 = qY_0$. Substituting these values in equation (19) we have

$$\rho(X_0) \cong (c+1)X_0 - \Phi\left(\frac{pX_0-qY_0}{\sqrt{qY_0}}\right)[v + X_0 - \frac{q}{p} Y_0]$$
$$- \sqrt{\frac{qY_0}{p}}\,\phi\left(\frac{pX_0 - qY_0}{\sqrt{qY_0}}\right). \tag{20}$$

Under the same sort of assumptions employed in Section 4, this risk function may be differentiated, and the optimal X_0 value obtained numerically. This approach was employed for the same cases used in standard central limit theorem approach of section 4. The results are presented in Table 2. Table 3 presents a comparison of the results obtained from these two methods. As can be seen, the agreement of the methods, both in terms of optimal force level and optimal risk, is quite good.

Table 2 - Numerical Results for the One Stage Decision Problem Martingale Method

Notation: Y_0 - Initial enemy force level

p - Probability next casualty is enemy

c - Cost of employing friendly troops

v - Reward for victory (totally destroying enemy force)

X_0 - Optimal initial force level

$\rho(X_0)$ - Risk of optimal force level

y	p	c	v	X_0	$\rho(X_0)$
100	0.5	0.5	500	136.55	-329.31
1000	0.5	0.5	5000	1134.03	-3426.19
100	0.3	0.5	500	297.66	-112.67
1000	0.3	0.5	5000	2576.84	-1363.92
100	0.7	0.5	500	64.80	-423.49
1000	0.7	0.5	5000	507.46	-4314.12
100	0.5	0.6	500	135.54	-315.71
1000	0.5	0.6	5000	1131.28	-3312.93
100	0.3	0.6	500	295.42	-83.04
1000	0.3	0.6	5000	2570.94	-1106.56
100	0.7	0.6	500	64.28	-417.04
1000	0.7	0.6	5000	506.03	-4263.45

Table 3 - Comparison of Martingale and Standard Central Limit Theorem Results

case	Standard $(X_0,\rho(X_0))$	Martingale $(X_0,\rho(X_0))$
1	136.94,-328.89	136.55,-329.31
2	1134.40,-3426.12	1134.03,-3426.19
3	299.14,-111.95	297.66,-112.67
4	2578.36,-1363.33	2576.84,-1363.92
5	64.92,-423.43	64.80,-423.49
6	507.58,-4314.08	507.46,-4314.12
7	135.92,-319.45	135.54,-315.71
8	1131.67,-3312.00	1131.28,-3312.93
9	296.90,-82.16	295.42,-83.04
10	2572.47,-1105.37	2570.94,-1106.56
11	64.40,-416.94	64.28,-417.04
12	506.16,-4263.16	506.03,-4263.45

The case of the square law is similar to the linear law but differs in detail and ease of solution. In section 5, the $P(X_f>0)$ and $E(X_f|X_f>0)$ where computed approximately. The risk function is given by

$$\rho(X_0) = (c+1)X_0 - E(X_f|X_0,X_f>0)P(X_f>0) - vP(X_f>0).$$

thus

$$\rho(X_0) \doteq (c+1)X_0 + \frac{\Phi(\mu/\sigma)}{2}\{1+2v-(\int_0^\infty (1+\frac{8X}{b})^{1/2}\varphi(\frac{X-\mu}{\sigma})dx/\Phi(\mu/\sigma))\} \quad (21)$$

where μ and σ^2 are defined in (17) and (18).

The minimization of (21) is made difficult by the presence of the integral and that both μ and σ are functions of x_0. Consequently, the optimal x_0 is best found through a computer search. This task may be simplified by recognizing that $E(X_f|X_0,X_f>0)$ can be represented by $\frac{1}{2}E(T^{1/2}|T>1) - 1/2$ where T is a random variable with a normal distribution. Since $T = 1 + \frac{8K_f}{b}$, the mean and variance of its normal distribution are easily determined from K_f.

7. Summary and Suggestions for Further Research

This paper has formulated a framework for an elementary one stage decision problem in a simplified combat environment. The combat models employed were based on stochastic versions of the classical Lanchester attrition laws.

The risk function to be minimized involved a victory probability and the expected number of survivors. These quantities were obtained approximately by using either the usual central limit theorem or the martingale central limit theorem applied to the specific attrition structure. Numerical solutions seem to generally be required.

The ideas and results presented in this paper represent a first step in applying the concepts of statistical decision theory to Lanchester processes. In this paper, only the simplest possible decision problem was addressed. In a further paper, the diffusion approximation methodology presented in Perla and Lehoczky [6] will be used to study multi-stage problems. In such cases, the decision maker will have a series of force allocation decisions to make as opportunities for reinforcement or withdrawal present themselves. The solution to such multi-stage problems, as well as other possible problems of interest, will be based, to a large extent, on the methodology presented here.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Decision Theoretic Approach to Lanchester Processes: I. The One Period Problem		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER TR #146
7. AUTHOR(s) Peter P. Perla Center for Naval Analyses		8. CONTRACT OR GRANT NUMBER(s) N00014-77-C-D161 (NR 277-227)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Carnegie-Mellon University Pittsburgh, PA 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 277-227
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Analysis Programs Code 431 Office of Naval Research Arlington, VA		12. REPORT DATE April, 1978
		13. NUMBER OF PAGES 34
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lanchester Theory, Stochastic Models, Optimal Decision Making, Martingale and Diffusion Approximations.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

See Report

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